View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by CERN Document Server

arXiv:quant-ph/0703011v1 1 Mar 2007

Stochastic particle annihilation: a relativistic model of quantum state reduction

Daniel J. Bedingham*

Blackett Laboratory, Imperial College, London SW7 2BZ, UK

(Dated: March 2, 2007)

A relativistic model of quantum state reduction involving a nonlinear stochastic extension of Schrödinger's equation is outlined. The eigenstates of the annihilation operator are chosen as the preferred basis onto which reduction occurs. These are the coherent states which saturate the bound of the Heisenberg uncertainty relation, exhibiting classical-like behavior. The quantum harmonic oscillator is studied in detail before generalizing to relativistic scalar quantum field theory. The infinite rates of increase in energy density which have plagued recent relativistic proposals of dynamical state reduction are absent in this model. This is because the state evolution equation does not drive particle creation from the vacuum. It is demonstrated how state reduction to a charge density basis can be induced in fermionic matter via an appropriate coupling to a bosonic field undergoing this mechanism.

PACS numbers: 03.65.Ta, 11.10.-z

I. INTRODUCTION

Much of the peculiar behavior associated with quantum physics results from the fact that, although a quantum system can be in a superposition of different states, whenever we make measurements involving macroscopic apparatus, a definite state is always registered. The transition from a superposition to a definite state is not described by Schrödinger's equation. How then, if the constituents of the apparatus are also described by Schrödinger's equation, does this quantum state reduction come about?

Stochastic generalizations of Schrödinger's equation have been proposed by a number of authors in answer to the problem of measurement [1, 2, 3, 4, 5] (for a review see [6, 7]). The key idea is that measurement is understood as the realization of a random process in the Hilbert space of state vectors where unwanted superpositions of states are unstable. The appeal of these models rests on two fundamental properties: (i) they reproduce quantum effects on small scales with negligible modification to standard quantum theory, and (ii) they lead to rapid, objective state vector collapse on large scales with probabilities given by the laws of standard quantum mechanics. The result is that superpositions of states for macroscopic objects are suppressed whilst individual particles continue to behave according to quantum theory.

The usual approach is to substitute Schrödinger's equation with a quantum state diffusion equation of the form

$$\mathrm{d}|\psi_t\rangle = \left(C\mathrm{d}t + \mathbf{A}\cdot\mathrm{d}\mathbf{X}_t\right)|\psi_t\rangle. \tag{1}$$

*Electronic address: d.bedingham@imperial.ac.uk

Here $\{\mathbf{X}_t\}$ is a (vector-valued) Itô process and $\mathbf{A}$, C are operators (the Schrödinger equation can be recovered by setting $C = -\mathrm{i}H$ and $\mathbf{A} = 0$). With appropriate choices for the drift and volatility of $\{\mathbf{X}_t\}$ the quantum state typically evolves into an eigenstate of the operator $\mathbf{A}$. The choice of $\mathbf{A}$ leads to a preferred basis. In the quantum mechanical case, the standard choice is a locally averaged position state basis in order to reproduce the definite localization of objects at the classical scale. Another idea is to use an energy state basis [8, 9, 10]. These models have the desirable property that energy is conserved in expectation. A general solution to the energy-based state diffusion with time-dependent coupling has recently been found [11].

At present, non-relativistic proposals are seen to have sufficiently negligible effects on the quantum scale in order to be indistinguishable from standard quantum theory for current experimental technologies [12]. At the same time these proposals offer a consistent understanding of classical and quantum domains. However, so far, relativistic field theoretic formulations generally predict an infinite rate of particle creation due to the coupling of a classical stochastic field to a quantum scalar field [6, 13, 14, 15]. Some previous attempts to resolve this problem have involved either modifying the stochastic field to prevent high-energy excitations [7, 16], or coupling the noise source not locally to the quantum field but to the integral of quantum fields over some space-time region [17]. A quantum mechanical model for a relativistic particle has been developed in reference [18] although this model does not include interactions.

In this paper we outline an alternative proposal in which the stochastic field is coupled only to the annihilation operators of the quantum scalar field (via a local interaction term). The scalar field cannot then be excited by the stochastic field. As a consequence the infinite rates of energy increase are avoided. Instead we see an expected energy loss to the stochastic field which can be controlled to a negligibly small level by an appropriate choice for the coupling parameter. A similar idea has been considered before by Pearle [19] although his model is not Lorentz invariant.

We will find that the quantum state evolves towards the eigenstates of the annihilation operators. In quantum mechanics these are well understood as coherent states (see e.g. ref. [20]). The coherent states have long been regarded as a close quantum approximation to idealized classical states and therefore constitute a natural choice for the preferred basis states in a quantum state reduction model.

The paper is organized as follows. In section II we demonstrate the state reduction mechanism for the simple case of a quantum harmonic oscillator. By analyzing the quantum variance processes we are able to demonstrate that state reduction occurs, and to estimate the associated reduction timescale. We also examine how the expectation of energy evolves and demonstrate that initial quantum probabilities match with the probabilities of stochastic outcomes in a simple example. We conclude the section with some numerical results which confirm our analysis.

In section III we extend the formalism to a relativistic quantum scalar field. We adopt the interaction picture of Tomonaga and Schwinger [21, 22] to describe a state defined on some space-like hyper-surface evolving in a time-like manner. Once we have established that our formulation is relativistically covariant, we proceed to demonstrate the reductive properties in a specific frame. We show how this mechanism of state reduction for a bosonic field could induce a state reduction to some charge state basis in a fermionic field. We end in section IV with some concluding remarks.

II. QUANTUM MECHANICAL HARMONIC OSCILLATOR

The device we shall use to represent quantum state reduction will be presented for the case of $(0+1)$-dimensional scalar field theory, i.e. the quantum mechanical harmonic oscillator. The commutation relation between position and momentum is given by $[x,p]=\mathrm{i}$. We define creation and annihilation operators in the standard way as follows

$$\begin{cases} a = \sqrt{\frac{\omega}{2}}(x+\mathrm{i}p\omega^{-1}) \\ a^\dagger = \sqrt{\frac{\omega}{2}}(x-\mathrm{i}p\omega^{-1}) \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = \frac{1}{\sqrt{2\omega}}(a+a^\dagger) \\ p = -\mathrm{i}\sqrt{\frac{\omega}{2}}(a-a^\dagger) \end{cases} \tag{2}$$

These operators satisfy the commutation relation $[a,a^\dagger]=1$. The Hamiltonian for the harmonic oscillator is given by

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 x^2 = \omega(a^\dagger a + \tfrac{1}{2}) = \omega(N+\tfrac{1}{2}), \tag{3}$$

where $N=a^\dagger a$ is the particle number operator. Units are chosen such that $\hbar=1$ for the sake of simplicity.

The Schrödinger equation expressed in differential form is $\mathrm{d}|\psi_t\rangle = -\mathrm{i}H|\psi_t\rangle\mathrm{d}t$. We extend this in the following way

$$\mathrm{d}|\psi_t\rangle = \left\{\left[-\mathrm{i}H - \tfrac{1}{2}\lambda^2(a^\dagger-\bar{a}_t)a + \tfrac{1}{2}\lambda^2(a-\bar{a}_t)\bar{a}_t\right]\mathrm{d}t + \lambda(a-\bar{a}_t)\mathrm{d}B_t\right\}|\psi_t\rangle, \tag{4}$$

where

$$\bar{a}_t = \tfrac{1}{2}\langle\psi_t|(a+a^\dagger)|\psi_t\rangle, \tag{5}$$

and λ is a constant parameter of dimension $[time]^{-1/2}$. Denoting unconditional expectation with respect to the physical probability measure $\mathbb{P}$ by $\mathbb{E}^{\mathbb{P}}[\cdot]$, the differential $\mathrm{d}B_t$ is an increment of real $\mathbb{P}$-Brownian motion with the properties that increments at different times are independent, $\mathbb{E}^{\mathbb{P}}[\mathrm{d}B_t]=0$, and $(\mathrm{d}B_t)^2=\mathrm{d}t$. Equation (4) can be derived (see [6]) by first assuming a state evolution equation of the form $\mathrm{d}|\phi_t\rangle = (C\mathrm{d}t + \lambda a\mathrm{d}X_t)\,|\phi_t\rangle$ where $|\psi_t\rangle = |\phi_t\rangle/\langle\phi_t|\phi_t\rangle^{1/2}$ and where $\{X_t\}$ is a $\mathbb{Q}$-Brownian motion. The physical measure $\mathbb{P}$ is related to $\mathbb{Q}$ through $\mathbb{P}(A)=\mathbb{E}^{\mathbb{P}}[\mathbf{1}_A]=\mathbb{E}^{\mathbb{Q}}[\langle\phi_t|\phi_t\rangle\mathbf{1}_A]$ for some event A measurable at time t, where $\mathbf{1}_A=1$ if A is true and 0 otherwise. This choice of physical probability measure is the counterpart to the postulate of standard quantum mechanics on the outcomes of measurement processes [6].

Note that since the state evolves according to equation (4) by the action of only the number operator and the annihilation operator, a final state with higher energy than any of those states contributing to the initial superposition $|\psi_0\rangle$ cannot occur. This ensures that as long as the initial state has finite energy, subsequent evolved states must also have finite energy.

We proceed by demonstrating that equation (4) preserves the norm of a state. Denoting $|\mathrm{d}\psi_t\rangle = \mathrm{d}|\psi_t\rangle$ we have

$$\begin{aligned} \mathrm{d}(\langle\psi_t|\psi_t\rangle) &= \langle\mathrm{d}\psi_t|\psi_t\rangle + \langle\psi_t|\mathrm{d}\psi_t\rangle + \langle\mathrm{d}\psi_t|\mathrm{d}\psi_t\rangle \\ &= \langle\psi_t|\left[\mathrm{i}H - \tfrac{1}{2}\lambda^2 a^\dagger(a-\bar{a}_t) + \tfrac{1}{2}\lambda^2(a^\dagger-\bar{a}_t)\bar{a}_t\right]|\psi_t\rangle\mathrm{d}t + \langle\psi_t|\lambda(a^\dagger-\bar{a}_t)|\psi_t\rangle\mathrm{d}B_t \\ &\quad + \langle\psi_t|\left[-\mathrm{i}H - \tfrac{1}{2}\lambda^2(a^\dagger-\bar{a}_t)a + \tfrac{1}{2}\lambda^2(a-\bar{a}_t)\bar{a}_t\right]|\psi_t\rangle\mathrm{d}t + \langle\psi_t|\lambda(a-\bar{a}_t)|\psi_t\rangle\mathrm{d}B_t \\ &\quad + \langle\psi_t|\lambda^2(a^\dagger-\bar{a}_t)(a-\bar{a}_t)|\psi_t\rangle\mathrm{d}t \\ &= 0. \end{aligned} \tag{6}$$

For convenience we take the norm of the initial state $|\psi_0\rangle$ to be unity. Further, we make the following definitions for the conditional expectation and conditional variance of some operator O with respect to the state $|\psi_t\rangle$ at time t

$$O_t = \langle\psi_t|O|\psi_t\rangle \quad \text{and} \quad V_t^O = \langle\psi_t|(O^\dagger - O_t^*)(O - O_t)|\psi_t\rangle,$$

and the conditional covariance of two operators O and O'

$$V_t^{O,O'} = \langle\psi_t|(O^\dagger - O_t^*)(O' - O'_t)|\psi_t\rangle.$$

In addition, we define the operator $\Delta O_t = O - O_t$.

Let us first consider the energy of the oscillator. It is straightforward to demonstrate that the energy process $H_t = \langle\psi_t|H|\psi_t\rangle$ satisfies the evolution equation

$$\mathrm{d}H_t = -\lambda^2\omega N_t\mathrm{d}t + \lambda\omega\langle\psi_t|a^\dagger a^\dagger a + a^\dagger aa - 2a^\dagger a\bar{a}_t|\psi_t\rangle\mathrm{d}B_t. \tag{7}$$

By taking the unconditional expectation we infer that

$$\mathbb{E}^{\mathbb{P}}[H_t] = H_0 - \lambda^2\mathbb{E}^{\mathbb{P}}\left[\int_0^t \mathrm{d}u\omega N_u\right] = H_0 - \lambda^2\omega\int_0^t \mathrm{d}u\mathbb{E}^{\mathbb{P}}\left[N_u\right]. \tag{8}$$

The second term on the right side is negative semi-definite. Therefore, energy is lost from the harmonic oscillator on average at a rate determined by the number of excitations. We demand that energy loss on a macroscopic scale is negligible in order to conform with the energy conservation principle. Taking the typical particle number in the state $|\psi_t\rangle$ to be of order N_0, we therefore require that $\lambda^2\omega N_0\Delta t \ll H_0$ for typical timescales Δt. Equivalently we may say that λ must be very small in standard macroscopic units of time. In this limit we have that $\mathbb{E}^{\mathbb{P}}[H_t] \simeq H_0$, or that the expected energy is approximately conserved. In addition, having very small λ means that for a small number of particles, equation (4) can be accurately approximated by Schrödinger's equation.

A. State reduction

In order to see how the collapse mechanism works we consider the stochastic processes a_t and V_t^a for the conditional expectation of the annihilation operator and the associated conditional variance:

$$\begin{aligned}
\mathrm{d}a_t &= -\mathrm{i}\omega a_t\mathrm{d}t - \tfrac{1}{2}\lambda^2 a_t\mathrm{d}t + \lambda\langle\psi_t|(a+a^\dagger)a - 2\bar{a}_t a|\psi_t\rangle\mathrm{d}B_t, && (9)\\
\mathrm{d}V_t^a &= -\lambda^2\left\{\langle\psi_t||\Delta a_t|^2|\psi_t\rangle + |\langle\psi_t|(a+a^\dagger)a - 2\bar{a}_t a|\psi_t\rangle|^2\right\}\mathrm{d}t \\
&\quad + \lambda\langle\psi_t|(a^\dagger - \bar{a}_t)|\Delta a_t|^2 + |\Delta a_t|^2(a - \bar{a}_t)|\psi_t\rangle\mathrm{d}B_t. && (10)
\end{aligned}$$

Taking the unconditional expectation of equation (10) we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[V_t^a] &= V_0^a - \lambda^2\mathbb{E}^{\mathbb{P}}\left[\int_0^t \mathrm{d}uV_u^a\right] - \lambda^2\mathbb{E}^{\mathbb{P}}\left[\int_0^t \mathrm{d}u|V_u^{(a+a^\dagger),a}|^2\right] \\
&= V_0^a - \lambda^2\int_0^t \mathrm{d}u\mathbb{E}^{\mathbb{P}}\left[V_u^a\right] - \lambda^2\int_0^t \mathrm{d}u\mathbb{E}^{\mathbb{P}}\left[|V_u^{(a+a^\dagger),a}|^2\right]. && (11)
\end{aligned}$$

Since the last two terms on the right side are positive semi-definite, the unconditional expectation of the variance of a cannot increase (i.e. V_t^a is a super-martingale). If we suppose that these terms are nonzero then $\mathbb{E}^{\mathbb{P}}[V_t^a] \to 0$ for large times and therefore $V_t^a \to 0$ i.e. the state enters an a-eigenstate. Otherwise, if for some time t we have $\mathbb{E}^{\mathbb{P}}\left[V_t^a\right] = 0$ and $\mathbb{E}^{\mathbb{P}}\left[|V_t^{(a+a^\dagger),a}|^2\right] = 0$, then $|\psi_t\rangle$ at that time must be an a-eigenstate. Note that the second of these two conditions is also satisfied when $|\psi_t\rangle$ is a position eigenstate at time t. Since these are composed of an infinite number of infinitesimal energy mode contributions, we exclude this possibility.

In order to estimate the characteristic timescale for state reduction we approximate equation (11) by freezing the stochastic terms on the right side at $t = 0$. In this approximation we find

$$\frac{\mathbb{E}^{\mathbb{P}}[V_t^a] - V_0^a}{V_0^a} \simeq -\lambda^2 \left(1 + \frac{|V_0^{(a+a^\dagger),a}|^2}{V_0^a}\right) t. \tag{12}$$

Taking $V_0^a \sim V_0^{(a+a^\dagger),a} \sim \mathcal{O}(N_0)$ (corresponding, for example, to a superposition between a large excited state and the vacuum state), the reduction timescale for the variance-decreasing process can be estimated as

$$\tau_R \sim \frac{V_0^a}{\lambda^2 |V_0^{(a+a^\dagger),a}|^2} \sim \frac{1}{\lambda^2 N_0}. \tag{13}$$

This must be small in standard units for macroscopic objects such that macroscopic superpositions are suppressed. For example, for an oscillator with frequency of order $10^{14}\mathrm{s}^{-1}$ (corresponding to visible light), if we take $N_0 = 10^{23}$ and $\hbar = 10^{-34}\mathrm{Js}$, then choosing $\lambda = 10^{-8} s^{-1/2}$ would lead to energy loss at a rate of $10^{-13}\mathrm{Js}^{-1}$ and state reduction on a timescale of order $10^{-7}\mathrm{s}$. For one particle ($N_0 = 1$) energy loss is of order $10^{-36}\mathrm{Js}^{-1}$ and the reduction timescale is $10^{16}\mathrm{s}$ (10^9 yrs).

Once the system enters an a-eigenstate, equation (9) reduces to

$$\mathrm{d}a_t = \left(-\mathrm{i}\omega - \tfrac{1}{2}\lambda^2\right) a_t \mathrm{d}t, \tag{14}$$

with solution $a_t = a_0 \exp\{-\mathrm{i}\omega t - \frac{1}{2}\lambda^2 t\}$. The solution decays on timescale λ^{-2} which as stated earlier must be very large in standard macroscopic units of time.

So far we have demonstrated that our state evolution equation (4) describes state reduction to a coherent state on timescale τ_R given in equation (13). We have also shown that coherent states themselves will decay to the vacuum state on a very long timescale λ^{-2}. We conclude this subsection by confirming that stochastic probabilities match with quantum probabilities for the outcome of a simplified measurement. Let us consider the projection operator of a particle number eigenstate $P_n = |n\rangle\langle n|$. The conditional expectation of the projection operator $(P_n)_t = \langle\psi_t|P_n|\psi_t\rangle$ obeys the evolution equation

$$\mathrm{d}(P_n)_t = \lambda^2\left[(n+1)(P_{n+1})_t - n(P_n)_t\right]\mathrm{d}t + \lambda\langle\psi_t|a^\dagger P_n + P_n a - 2\bar{a}_t P_n|\psi_t\rangle \mathrm{d}B_t, \tag{15}$$

where the terms in square brackets on the right side corresponds to the background decay mechanism occurring on timescale λ^{-2}. These terms together are small when a given wavepacket is sufficiently smoothly varying in n. (For example, a wavepacket centered at $n = n'$ with a standard deviation in n of $\mathcal{O}(\sqrt{n'})$, typically has $(P_n)_t \sim \mathcal{O}(1/\sqrt{n'})$ and

$[(P_{n+1})_t - (P_n)_t] \sim \mathcal{O}(1/n')$, resulting in $[(n+1)(P_{n+1})_t - n(P_n)_t] \sim \mathcal{O}(1)$. These orders of magnitude correspond to a minimum uncertainty coherent state wavepacket.)

Consider now an initial superposition state $|\psi_0\rangle$ consisting of the vacuum state $|0\rangle$ and some excited coherent state $|\alpha_0\rangle$. Suppose further that $\langle 0|\alpha_t\rangle \simeq 0$. We may think of this situation as corresponding to a superposition of null and positive readings on some measuring device.

After some time t where $\tau_R < t \ll \lambda^{-2}$ reduction has occurred to a coherent state. This may be either the vacuum state or $|\alpha_t\rangle$. The initial quantum probability for registering the system in the vacuum state is $(P_{\rm vac})_0 = \langle\psi_0|P_{\rm vac}|\psi_0\rangle$ where $P_{\rm vac} = |0\rangle\langle 0|$. From equation (15) we have (upon ignoring the terms in square brackets)

$$\mathrm{d}(P_{\rm vac})_t \simeq \lambda\langle\psi_t|a^\dagger P_{\rm vac} + P_{\rm vac}a - 2\bar{a}_t P_{\rm vac}|\psi_t\rangle \mathrm{d}B_t. \tag{16}$$

Now taking the unconditional expectation we have

$$(P_{\rm vac})_0 \simeq \mathbb{E}^{\mathbb{P}}\left[(P_{\rm vac})_t\right] \simeq \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{|\psi_t\rangle=|0\rangle}\right]. \tag{17}$$

The final approximation results from the fact that the state at time t is either the vacuum state or the approximately orthogonal excited coherent state $|\alpha_t\rangle$. This relation tells us that the initial standard quantum estimate for the probability of finding the system in the vacuum state is equal to the stochastic probability of that outcome occurring in this model. The quantum and stochastic probabilities for the other outcome must also be equal.

B. Numerical simulations

In order to confirm the reductive properties we ran a numerical simulation of the quantum state evolution. We considered an initial state corresponding to an equal superposition of two a-eigenstates with eigenvalues 0 and 8 respectively. We have set the parameters to $\lambda = 0.5$ and $\omega = 1$. This choice means we observe state reduction for small numbers of particles with only a small degree of energy loss. Since $N_0 \sim 32$ we estimate the reduction timescale by equation (13) to be $\tau_R \sim 0.125$. The decay timescale is given by $\lambda^{-2} \sim 4$. These order-of-magnitude estimates are confirmed by figures 1 and 2 which show sample paths for the conditional expectation of energy and for the conditional variance in a respectively. We see that the state evolves into either one of the two possible coherent states. One of these states is the vacuum state, the other corresponds to the (slowly decaying) non-vacuum coherent state.

In addition we have estimated the probabilities of the two possible outcomes by running 100 sample paths. We find probabilities of 0.47 for the vacuum state and 0.53 for the non-vacuum state (the standard deviation of this estimate is 0.1).

III. RELATIVISTIC QUANTUM FIELD THEORY

Here we generalize the previous section to the case of relativistic quantum field theory. (For a discussion of the conceptual issues surrounding the formulation of a relativistic state reduction model, see [14, 23, 24, 25].) Given that experimental evidence conforms to the principle of relativistic invariance it is natural to require it of our model. This has been a longstanding problem in the field of dynamical state reduction models. The reason is that

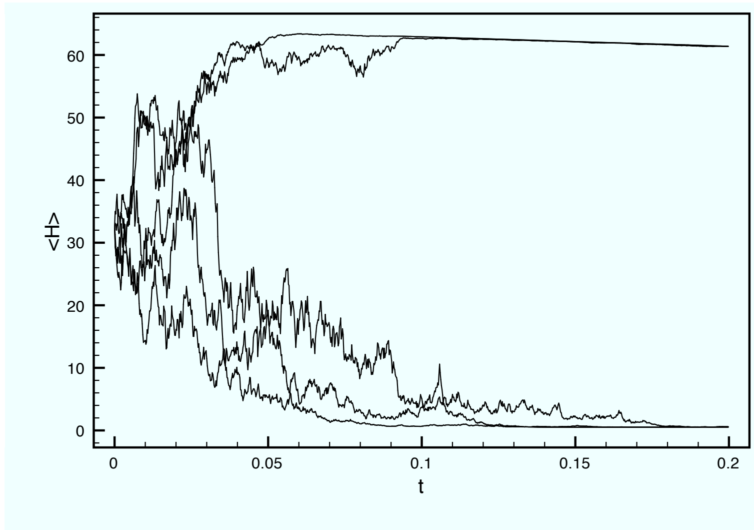

FIG. 1: Conditional expectation of energy. The plot shows five realized paths for an initial state corresponding to an equal superposition of two coherent states with expected energies 0.5 and 64.5 respectively. In the cases where the state reduces to the excited coherent state we note a slow decay in energy. This is expected to occur on a timescale of order $\lambda^{-2} \sim 4$ in this example ($\lambda = 0.5$ and $\omega = 1$).

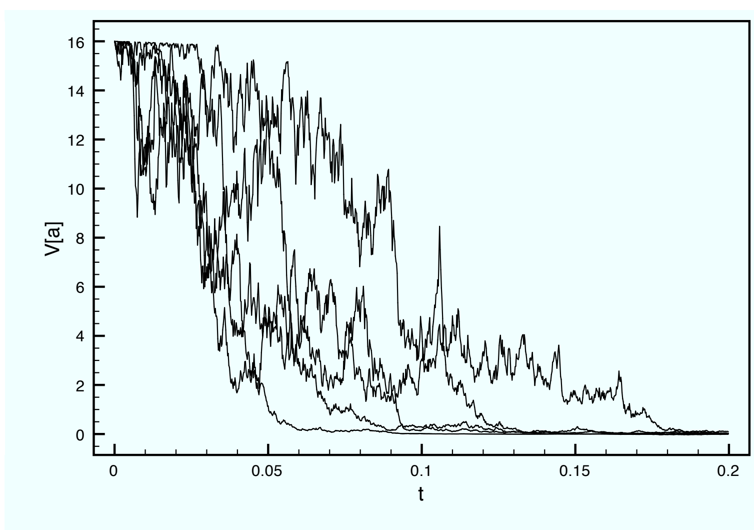

FIG. 2: Conditional variance of the annihilation operator. The sample paths correspond to those in figure 1 ($\lambda = 0.5$ and $\omega = 1$).

while state reduction can be modelled easily enough, by coupling a stochastic process to a quantum field we generate energy at an infinite rate. We will resolve this issue by coupling only the annihilation operators of the quantum field to the stochastic process (as in the case of the harmonic oscillator discussed in the previous section). This will ensure that energy cannot be created from the vacuum.

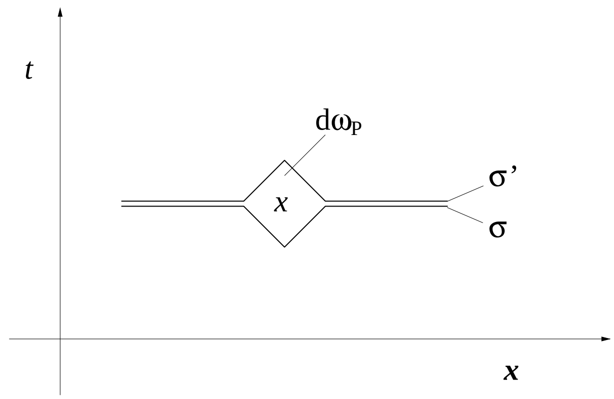

FIG. 3: Evolution between space-like hyper-surfaces σ and σ'.

A natural formulation of relativistic quantum field theory for the consideration of an evolving state is the one due to Tomonaga and Schwinger [13, 21, 22]. We write the Hamiltonian density at space-time point x in the form $H(x) = H_0(x) + H_{\mathrm{int}}(x)$, where H_0 is the free field Hamiltonian and H_{int} is an interaction term. Then evolution of the quantum state is described by the Tomonaga equation:

$$\mathrm{i}\frac{\delta}{\delta\sigma(x)}|\Psi(\sigma)\rangle = H_{\mathrm{int}}(x)|\Psi(\sigma)\rangle. \tag{18}$$

The state is defined on some space-like three-surface σ. Functional differentiation is defined with respect to some point x lying on σ. Given two space-like surfaces σ and σ' differing only by some infinitesimal spacetime volume $\mathrm{d}\omega_x$ at point x (see figure 3) the functional derivative can be expressed as

$$\frac{\delta|\Psi(\sigma)\rangle}{\delta\sigma(x)} = \lim_{\sigma'\to\sigma}\frac{|\Psi(\sigma')\rangle - |\Psi(\sigma)\rangle}{\mathrm{d}\omega_x}. \tag{19}$$

Equation (18) describes the evolution of the quantum state in terms of incremental time-like advancements of individual points on a space-like surface. The operator H_{int} must be a scalar quantity in order that equation (18) has a relativistically invariant form. In addition, it must commute with itself between all pairs of points on a given space-like surface so that the ordering of points undergoing time-like advancement is irrelevant.

In differential form the Tomonaga equation can be represented as follows

$$\mathrm{d}_x|\Psi(\sigma)\rangle = -\mathrm{i}H_{\mathrm{int}}(x)|\Psi(\sigma)\rangle\mathrm{d}\omega_x. \tag{20}$$

We proceed by generalizing this equation to a diffusion equation.

A. Field state diffusion equation

Previous approaches to modifying Schrödinger field dynamics have generally involved the inclusion of a white-noise field term in the Tomonaga equation (see e.g. [6]). Here we opt to formulate our model in terms of a Brownian process. This will help us to understand the nonlocal correlations in the stochastic information necessary for a consistent model. We

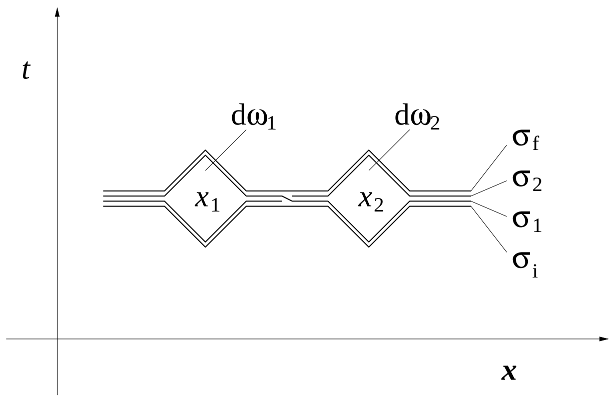

FIG. 4: Evolution from space-like hyper-surface σ_i at points x_1 and x_2. The final state at σ_f should be independent of the ordering $\sigma_i \to \sigma_1 \to \sigma_f$ or $\sigma_i \to \sigma_2 \to \sigma_f$.

begin by defining $\mathrm{d}B_x$ to be an increment of some real $\mathbb{P}$-Brownian motion with mean zero and covariance given by

$$\mathbb{E}^{\mathbb{P}}[\mathrm{d}B_x \mathrm{d}B_{x'}] = \delta_{x,x'}\mathrm{d}\omega_x. \tag{21}$$

We may think of the Gaussian random variable $B(\sigma)$ defined on some surface σ and of $\mathrm{d}B_x$ as the incremental difference in B between two surfaces differing by some infinitesimal space-time volume at point x. We will further examine this notion throughout this subsection.

We extend the differential Tomonaga equation to include a stochastic term as follows

$$\mathrm{d}_x|\Psi(\sigma)\rangle = \big(\alpha(x,\sigma)\mathrm{d}\omega_x + \beta(x,\sigma)\mathrm{d}B_x\big)|\Psi(\sigma)\rangle. \tag{22}$$

A similar equation can be found in ref. [15]. Note that we have allowed the quantum operators α and β to depend not only on the local space-time point x but on the surface σ over which the initial quantum state is defined. This is natural if α and β are to involve nonlinear terms as in equation (4). We initially choose a definite linear ordering for the time-like advancements of individual points on our surface. The generalized Gaussian process B is then no different from the standard Brownian motion. We argue that the linear ordering is irrelevant and that any causal choice (whereby $\sigma_f > \sigma_i$ if σ_f is nowhere in the past of any point on σ_i) will give the same outcome.

Consider a process driven by some scalar operator $A(x)$ which is not necessarily self-adjoint but where, for two different points x and x', $[A(x), A(x')] = [A(x), A^\dagger(x')] = 0$. We assume the following form for α and β (cf. equation (4)),

$$\alpha(x,\sigma) \;=\; -\tfrac{1}{2}\lambda^2\big(A^\dagger(x) - \langle\bar{A}(x)\rangle_\sigma\big)A(x) + \tfrac{1}{2}\lambda^2\big(A(x) - \langle\bar{A}(x)\rangle_\sigma\big)\langle\bar{A}(x)\rangle_\sigma \tag{23}$$

$$\beta(x,\sigma) \;=\; \lambda\big(A(x) - \langle\bar{A}(x)\rangle_\sigma\big), \tag{24}$$

where $\langle\cdot\rangle_\sigma = \langle\Psi(\sigma)|\cdot|\Psi(\sigma)\rangle$ and $\bar{A}(x) = \frac{1}{2}\big(A(x) + A^\dagger(x)\big)$. We further assume that $A(x)$ commutes with the free Hamiltonian $H_0(x')$ for $x \neq x'$.

Now consider two incremental time-like advances of the space-like surface on which our initial state is defined (see figure 4). One such advance occurs at x_1 whilst the other is at x_2. Each of these points lies on the initial surface. Points x_1 and x_2 are space-like separated from one another and the order in which we perform the two evolutions should not be meaningful in a covariant description. Suppose that the initial surface is σ_i. If we choose to first advance at point x_1 then we arrive at the intermediate surface σ_1 before then advancing point x_2 to finally arrive at the state σ_f. Alternatively we could first advance point x_2 where the intermediate surface is σ_2 before finally advancing point x_1 to arrive at σ_f. Let us consider the first scenario $\sigma_i \to \sigma_1 \to \sigma_f$. From equation (22) we have

$$\begin{aligned}
|\Psi(\sigma_f)\rangle_{\sigma_i\to\sigma_1\to\sigma_f} &= \big\{1+\alpha(x_2,\sigma_1)\mathrm{d}\omega_2+\beta(x_2,\sigma_1)\mathrm{d}B_2\big\} \\
&\quad\times\big\{1+\alpha(x_1,\sigma_i)\mathrm{d}\omega_1+\beta(x_1,\sigma_i)\mathrm{d}B_1\big\}|\Psi(\sigma_i)\rangle \\
&= \Big\{1+\alpha(x_1,\sigma_i)\mathrm{d}\omega_1+\alpha(x_2,\sigma_1)\mathrm{d}\omega_2+\beta(x_1,\sigma_i)\mathrm{d}B_1+\beta(x_2,\sigma_1)\mathrm{d}B_2 \\
&\quad+\alpha(x_2,\sigma_1)\alpha(x_1,\sigma_i)\mathrm{d}\omega_2\mathrm{d}\omega_1+\alpha(x_2,\sigma_1)\beta(x_1,\sigma_i)\mathrm{d}\omega_2\mathrm{d}B_1 \\
&\quad+\beta(x_2,\sigma_1)\alpha(x_1,\sigma_i)\mathrm{d}\omega_1\mathrm{d}B_2+\beta(x_2,\sigma_1)\beta(x_1,\sigma_i)\mathrm{d}B_2\mathrm{d}B_1\Big\}|\Psi(\sigma_i)\rangle, \quad (25)
\end{aligned}$$

where we have used the shorthand $\mathrm{d}\omega_{x_i} = \mathrm{d}\omega_i$ and $\mathrm{d}B_{x_i} = \mathrm{d}B_i$. We also define $A(x_i) = A_i$ and $\mathrm{d}_{x_i} = \mathrm{d}_i$. Given $\mathrm{d}B_1$ we can use equation (22) to find

$$\begin{aligned}
\mathrm{d}_1\langle\bar{A}_2\rangle_{\sigma_i} &= \big(\mathrm{d}_1\langle\Psi(\sigma_i)|\big)\bar{A}_2|\Psi(\sigma_i)\rangle+\langle\Psi(\sigma_i)|\bar{A}_2\big(\mathrm{d}_1|\Psi(\sigma_i)\rangle\big)+\big(\mathrm{d}_1\langle\Psi(\sigma_i)|\big)\bar{A}_2\big(\mathrm{d}_1|\Psi(\sigma_i)\rangle\big) \\
&= 2\lambda\big(\langle\bar{A}_1\bar{A}_2\rangle_{\sigma_i}-\langle\bar{A}_1\rangle_{\sigma_i}\langle\bar{A}_2\rangle_{\sigma_i}\big)\mathrm{d}B_1 \\
&= V_{1,2}\mathrm{d}B_1, \quad (26)
\end{aligned}$$

which we can use to derive $\mathrm{d}_1\beta(x_2,\sigma_i) = -\lambda V_{1,2}\mathrm{d}B_1$ and $\mathrm{d}_1\alpha(x_2,\sigma_i) = -\frac{1}{2}\lambda^2V_{1,2}^2\mathrm{d}\omega_1 + \lambda V_{1,2}\beta(2,\sigma_i)\mathrm{d}B_1$. These relations in turn imply that

$$\beta(x_2,\sigma_1) = \beta(x_2,\sigma_i) - \lambda V_{1,2}\mathrm{d}B_1, \quad (27)$$

and that

$$\alpha(x_2,\sigma_1) = \alpha(x_2,\sigma_i) - \tfrac{1}{2}\lambda^2V_{1,2}^2\mathrm{d}\omega_1 + \lambda V_{1,2}\beta(x_2,\sigma_i)\mathrm{d}B_1. \quad (28)$$

We now substitute equations (27) and (28) into equation (25) to obtain

$$\begin{aligned}
|\Psi(\sigma_f)\rangle_{\sigma_i\to\sigma_1\to\sigma_f} &= \Big\{1+\alpha(x_1,\sigma_i)\mathrm{d}\omega_1+\alpha(x_2,\sigma_i)\mathrm{d}\omega_2+\beta(x_1,\sigma_i)\mathrm{d}B_1+\beta(x_2,\sigma_i)\mathrm{d}B_2 \\
&\quad+\big[\alpha(x_2,\sigma_i)\alpha(x_1,\sigma_i)-\tfrac{1}{2}\lambda^2V_{1,2}^2+\lambda V_{1,2}\beta(x_2,\sigma_i)\beta(x_1,\sigma_i)\big]\,\mathrm{d}\omega_2\mathrm{d}\omega_1 \\
&\quad+\left[\beta(x_2,\sigma_i)\beta(x_1,\sigma_i)-\lambda V_{1,2}\right]\mathrm{d}B_2\mathrm{d}B_1 \\
&\quad+\alpha(x_2,\sigma_i)\beta(x_1,\sigma_i)\mathrm{d}\omega_2\mathrm{d}B_1+\beta(x_2,\sigma_i)\alpha(x_1,\sigma_i)\mathrm{d}\omega_1\mathrm{d}B_2 \\
&\quad+\lambda V_{1,2}\beta(x_2,\sigma_i)\mathrm{d}\omega_2\mathrm{d}B_1-\lambda V_{1,2}\beta(x_1,\sigma_i)\mathrm{d}\omega_1\mathrm{d}B_2\Big\}|\Psi(\sigma_i)\rangle. \quad (29)
\end{aligned}$$

Note that if we exclude the last two terms on the right side of equation (29), the remaining terms are symmetric under the interchange of 1 and 2 (recall that $[A(x), A(x')] = [A(x), A^\dagger(x')] = 0$ for $x \neq x'$). The last two terms are antisymmetric under $1 \leftrightarrow 2$. This means that in general

$$|\Psi(\sigma_f)\rangle_{\sigma_i\to\sigma_1\to\sigma_f} \neq |\Psi(\sigma_f)\rangle_{\sigma_i\to\sigma_2\to\sigma_f}. \quad (30)$$

However, if we introduce the transformation (effectively a rotation of the 2-D Brownian motion)

$$\begin{aligned}
\mathrm{d}B_1' &= \mathrm{d}B_1 - 2\lambda V_{1,2}\mathrm{d}\omega_1\mathrm{d}B_2 \\
\mathrm{d}B_2' &= \mathrm{d}B_2 + 2\lambda V_{1,2}\mathrm{d}\omega_2\mathrm{d}B_1 \quad (31)
\end{aligned}$$

and make explicit the dependence of the quantum state on the stochastic process, it is straightforward to show that

$$|\Psi(\sigma_f)\rangle_{\sigma_i\to\sigma_1\to\sigma_f}(\mathrm{d}B) = |\Psi(\sigma_f)\rangle_{\sigma_i\to\sigma_2\to\sigma_f}(\mathrm{d}B'). \tag{32}$$

The transformed increment of Brownian motion has the same properties as the original process. i.e. it has zero mean and $(\mathrm{d}B'_x)^2 = \mathrm{d}\omega_x$.

We can define a covariant Brownian motion $B(\sigma)$ as follows:

$$\begin{aligned}
\mathrm{d}B_1 &= B(\sigma_1) - B(\sigma_i), \\
\mathrm{d}B'_1 &= B(\sigma_f) - B(\sigma_2), \\
\mathrm{d}B_2 &= B(\sigma_f) - B(\sigma_1), \\
\mathrm{d}B'_2 &= B(\sigma_2) - B(\sigma_i).
\end{aligned} \tag{33}$$

The Brownian process $B(\sigma)$ is then uniquely defined on any given space-like surface σ. Defining the Brownian motion on the space-like surfaces in this way means that for given initial and final surfaces, we can evolve our quantum state in any order (respecting causality) and each time we achieve the same outcome. Although the state depends on the surface on which it is defined at any given point in its evolution, no preferred frame is associated with its past evolution. It is in this sense that the model is relativistically covariant.

These ideas are of particular relevance to the "Free Will Theorem" [26] which claims to show that relativistic dynamical reduction models are incompatible with the experimenter's free will to decide which observable to measure. In subsequent responses [27, 28] it has been argued that the resolution of this conflict can be found in nonlocality (see also [29]). Certainly equations (31) indicate nonlocal correlations in the stochastic information (in addition to the explicit nonlocal nature of equation (22)). However, at pointed out by 't Hooft [30], for models of this type we should reconsider our notion of "free will". For example, given some definite quantum state defined on some initial surface σ_i, and given some realized B for every space-like surface to the future of σ_i, then the quantum state on any future surface is certain. This future quantum state should describe all matter including the experimenter's free will.

B. Scalar field theory

Having established that it is possible to formulate the theory in a covariant form, we now focus on a particular frame with space-like surfaces chosen to be the constant time surfaces. We have

$$|\mathrm{d}\Psi(t)\rangle = \int_{\mathbf{x}} \mathrm{d}_x|\Psi(t)\rangle = \int_{\mathbf{x}} \mathrm{d}\mathbf{x}\big(\alpha(x)\mathrm{d}t + \beta(x)\mathrm{d}B_t(\mathbf{x})\big)|\Psi(t)\rangle, \tag{34}$$

with $\mathbb{E}^{\mathbb{P}}[\mathrm{d}B_t(\mathbf{x})\mathrm{d}B_{t'}(\mathbf{x}')] = \delta^3(\mathbf{x}-\mathbf{x}')\delta_{t,t'}\mathrm{d}t$. We use the integration subscript to avoid confusion over which variables are integrated over. In this frame, time-independent operators in the Schrödinger picture are related to time-dependent operators in the Tomonaga picture by the unitary transformation $O(t) = \exp\{\mathrm{i}H_0 t\}O\exp\{-\mathrm{i}H_0 t\}$, where H_0 is the free field Hamiltonian.

We consider a real scalar field φ defined in the Tomonaga picture by

$$\varphi(x) = \int \frac{\mathrm{d}\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}}\left\{\exp\left(\mathrm{i}\mathbf{p}\cdot\mathbf{x} - \mathrm{i}\omega_{\mathbf{p}}t\right)a(\mathbf{p}) + \exp\left(-\mathrm{i}\mathbf{p}\cdot\mathbf{x} + \mathrm{i}\omega_{\mathbf{p}}t\right)a^\dagger(\mathbf{p})\right\}, \tag{35}$$

with free Hamiltonian

$$
\begin{aligned}
H_0 &= \int \mathrm{d}\mathbf{x} \left\{ \tfrac{1}{2} \left(\partial_t \varphi(x)\right)^2 + \tfrac{1}{2} \nabla\varphi(x) \cdot \nabla\varphi(x) + \tfrac{1}{2} m^2 \varphi^2(x) \right\} \\
&= \int \mathrm{d}\mathbf{p} \omega_{\mathbf{p}} \left\{ a^\dagger(\mathbf{p}) a(\mathbf{p}) + \tfrac{1}{2}\delta^3(\mathbf{0}) \right\}, \qquad (36)
\end{aligned}
$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, and the creation and annihilation operators satisfy the canonical commutation relations $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta^3(\mathbf{p} - \mathbf{p}')$ and $[a(\mathbf{p}), a(\mathbf{p}')] = 0$. We now define new relativistic field operators by

$$
\Phi^+(x) = \int \mathrm{d}^4 p \exp\{\mathrm{i} p \cdot x\} \tilde{a}(p) \delta(0) \qquad (37)
$$

$$
\Phi^-(x) = \int \mathrm{d}^4 p \exp\{-\mathrm{i} p \cdot x\} \tilde{a}^\dagger(p) \delta(0) \qquad (38)
$$

with $\delta(0)[\tilde{a}(p), \tilde{a}^\dagger(p')] = \delta^4(p - p')$ and $[\tilde{a}(p), \tilde{a}(p')] = 0$. The operators $\tilde{a}(p)$ and $\tilde{a}^\dagger(p)$ are not restricted to excitations on mass shell. They are related to the usual creation and annihilation operators through

$$
\tilde{a}(p_0 = \omega_{\mathbf{p}}, \mathbf{p}) = \sqrt{2\omega_{\mathbf{p}}} a(\mathbf{p}) \quad \text{and} \quad \tilde{a}^\dagger(p_0 = \omega_{\mathbf{p}}, \mathbf{p}) = \sqrt{2\omega_{\mathbf{p}}} a^\dagger(\mathbf{p}). \qquad (39)
$$

These relations ensure that $\tilde{a}(p)$ and $\tilde{a}^\dagger(p)$ are scalar operators with respect to Lorentz transformations. The factor $\delta(0) = \delta(p_0^2 - \omega_{\mathbf{p}}^2)\theta(p_0)|_{p_0 = \omega_{\mathbf{p}}}$ is Lorentz invariant and ensures consistency with the on-shell commutation relations. By making this generalization we ensure that $[\alpha(x), \alpha(x')] = [\beta(x), \beta(x')] = 0$ and that $[\alpha(x), \beta(x')] = 0$ for $x \neq x'$ as can be checked given the following definition

$$
\alpha = -\tfrac{1}{2}\lambda^2 \left(\Phi^- - \tfrac{1}{2}\langle\Phi\rangle_t\right) \Phi^+ + \tfrac{1}{2}\lambda^2 \left(\Phi^+ - \tfrac{1}{2}\langle\Phi\rangle_t\right) \tfrac{1}{2}\langle\Phi\rangle_t \qquad (40)
$$

$$
\beta = \lambda \left(\Phi^+ - \tfrac{1}{2}\langle\Phi\rangle_t\right). \qquad (41)
$$

Here $\Phi = \Phi^+ + \Phi^-$ and $\langle\cdot\rangle_t = \langle\Psi(t)| \cdot |\Psi(t)\rangle$. We ignore for now any other possible Hamiltonian interaction terms. We will shortly show that if the initial state has no off-shell excitations then the number of excitations in off-shell modes will remain zero for future times.

In the same manner as (6) we can demonstrate that

$$
\mathrm{d}\langle\Psi(t)|\Psi(t)\rangle = 0, \qquad (42)
$$

so without loss of generality we may set $\langle\Psi(t)|\Psi(t)\rangle = 1$ with the state remaining normalized for all time.

Given some generic operator $O(t)$ in the Tomonaga picture, we may ask how its conditional expectation evolves. We find (cf. [15])

$$
\mathrm{d}\langle O\rangle_t = \langle \mathrm{d}O\rangle_t + \int_{\mathbf{x}} \mathrm{d}\mathbf{x} \langle \alpha^\dagger O + O\alpha + \beta^\dagger O \beta\rangle_t \mathrm{d}t + \int_{\mathbf{x}} \mathrm{d}\mathbf{x} \langle \beta^\dagger O + O\beta\rangle_t \mathrm{d}B_t(\mathbf{x}), \qquad (43)
$$

where dependencies on spatial coordinates are understood. The first term on the right side results from the standard unitary evolution of the operator O described by the free Hamiltonian.

Similarly we can write an evolution equation for the conditional variance of an operator. Recalling that $\Delta O_t = O - \langle O \rangle_t$ and that the conditional variance is given by $V_t[O] = \langle |\Delta O_t|^2 \rangle_t$, we find (again cf. [15])

$$\begin{aligned} \mathrm{d}V_t[O] &= \langle \mathrm{d}O^\dagger \Delta O_t + \Delta O_t^\dagger \mathrm{d}O \rangle_t + \int_{\mathbf{x}} \mathrm{d}\mathbf{x} \langle \alpha^\dagger |\Delta O_t|^2 + |\Delta O_t|^2 \alpha + \beta^\dagger |\Delta O_t|^2 \beta \rangle_t \mathrm{d}t \\ &\quad - \int_{\mathbf{x}} \mathrm{d}\mathbf{x} \langle \beta^\dagger O^\dagger + O^\dagger \beta \rangle_t \langle \beta^\dagger O + O \beta \rangle_t \mathrm{d}t \\ &\quad + \int_{\mathbf{x}} \mathrm{d}\mathbf{x} \langle \beta^\dagger |\Delta O_t|^2 + |\Delta O_t|^2 \beta \rangle_t \mathrm{d}B_t(\mathbf{x}). \end{aligned} \tag{44}$$

Note that the third term on the right side of equation (44) is negative semi-definite. This term is responsible for the variance reduction which we can use to demonstrate state reduction (see next subsection).

Consider the number of excitations in an off-shell mode p. The number operator is defined

$$\tilde{N}(p) = \tilde{a}^\dagger(p) \tilde{a}(p). \tag{45}$$

Applying equation (43) to the conditional expectation of $\tilde{N}(p)$ we have

$$\begin{aligned} \mathrm{d}\langle \tilde{N}(p) \rangle_t &= \int_{\mathbf{x}} \mathrm{d}\mathbf{x} \langle \alpha^\dagger \tilde{N}(p) + \tilde{N}(p) \alpha + \beta^\dagger \tilde{N}(p) \beta \rangle_t \mathrm{d}t \\ &\quad + \int_{\mathbf{x}} \mathrm{d}\mathbf{x} \langle \beta^\dagger \tilde{N}(p) + \tilde{N}(p) \beta \rangle_t \mathrm{d}B_t(\mathbf{x}). \end{aligned} \tag{46}$$

If we take an initial state at time t with no excitations in the off-shell p mode then we have

$$\begin{aligned} \mathrm{d}\langle \tilde{N}(p) \rangle_t &= \int_{\mathbf{x}} \mathrm{d}\mathbf{x} \langle \beta^\dagger \tilde{N}(p) \beta \rangle_t \mathrm{d}t \\ &= \int_{\mathbf{x}} \mathrm{d}\mathbf{x} \langle \beta^\dagger \beta \tilde{N}(p) - \lambda \exp\{\mathrm{i} p \cdot x\} \beta^\dagger \tilde{a}(p) \rangle_t \mathrm{d}t = 0. \end{aligned} \tag{47}$$

Therefore, given an initial state involving only on-shell excitations, subsequent evolution according to equation (34) cannot generate off-shell excitations.

We may also apply equation (43) to the total energy of the quantum field. Ignoring the vacuum energy and interactions, this is given by

$$H = \int \mathrm{d}\mathbf{p} \omega_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p}). \tag{48}$$

We find after some calculation that

$$\mathrm{d}\langle H \rangle_t = -\tfrac{1}{2} \lambda^2 \langle N \rangle_t \mathrm{d}t + \int_{\mathbf{x}} \mathrm{d}\mathbf{x} \langle \beta^\dagger H + H \beta \rangle_t \mathrm{d}B_t(\mathbf{x}), \tag{49}$$

where

$$N = \int \mathrm{d}\mathbf{p} a^\dagger(\mathbf{p}) a(\mathbf{p}). \tag{50}$$

Taking the unconditional expectation of the energy process at time t we have

$$\mathbb{E}^{\mathbb{P}}[\langle H\rangle_t] = \langle H\rangle_0 - \tfrac{1}{2}\lambda^2\int_0^t \mathrm{d}u\,\mathbb{E}^{\mathbb{P}}[\langle N\rangle_u], \tag{51}$$

(cf. equation (8)). Since $\langle N\rangle_t$ is nonnegative, it follows from (51) that energy is lost on average as a result of coupling the quantum field to a classical stochastic process. However, the energy loss is finite and can be made negligible by an appropriate choice of λ. This is to be contrasted with some of the previous attempts to construct a relativistic state reduction model [13, 14, 15], where the energy density is seen to increase at an infinite rate. The reason that we do not see an infinite rate of energy density creation can be traced back to the fact that the classical stochastic process is not coupled to the particle creation operator and therefore cannot drive particle creation from the vacuum.

Stochastic movements in the energy process will cease when the quantum state is an eigenstate of the operator Φ. When this occurs, the final term on the right side of equation (49) goes to zero.

C. Quantum field state reduction

To see the reductive properties we consider the particle annihilation operator $a(\mathbf{p})$. Using equation (43) we find

$$\mathrm{d}\langle a(\mathbf{p})\rangle_t = -\mathrm{i}\omega_{\mathbf{p}}\langle a(\mathbf{p})\rangle_t\mathrm{d}t - \frac{\lambda^2}{4\omega_{\mathbf{p}}}\langle a(\mathbf{p})\rangle_t\mathrm{d}t + \lambda\int_{\mathbf{x}}\mathrm{d}\mathbf{x}\langle\left(\Phi^+ + \Phi^- - \langle\Phi\rangle_t\right)a(\mathbf{p})\rangle_t\mathrm{d}B_t(\mathbf{x}). \tag{52}$$

Similarly using equation (44) and taking the unconditional expectation we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[V_t[a(\mathbf{p})]] &= V_0[a(\mathbf{p})] - \frac{\lambda^2}{2\omega_{\mathbf{p}}}\mathbb{E}^{\mathbb{P}}\left[\int_0^t \mathrm{d}u V_u[a(\mathbf{p})]\right] \\ &\quad -\lambda^2\mathbb{E}^{\mathbb{P}}\left[\int_0^t \mathrm{d}u\int_{\mathbf{x}}\mathrm{d}\mathbf{x}|\langle\left(\Phi^+ + \Phi^- - \langle\Phi\rangle_t\right)a(\mathbf{p})\rangle_t|^2\right] \\ &= V_0[a(\mathbf{p})] - \frac{\lambda^2}{2\omega_{\mathbf{p}}}\int_0^t \mathrm{d}u\,\mathbb{E}^{\mathbb{P}}\left[V_u[a(\mathbf{p})]\right] \\ &\quad -\lambda^2\int_0^t \mathrm{d}u\,\mathbb{E}^{\mathbb{P}}\left[\int_{\mathbf{x}}\mathrm{d}\mathbf{x}|\langle\left(\Phi^+ + \Phi^- - \langle\Phi\rangle_t\right)a(\mathbf{p})\rangle_t|^2\right].\end{aligned} \tag{53}$$

Again we find that the conditional variance for the annihilation operator is a supermartingale. The expected variance decreases with time and the quantum state evolves towards an eigenstate of the annihilation operator. We may estimate the timescale for collapse in the same manner as equations (12) and (13) by taking $V_0[a(\mathbf{p})] \sim N_0(\mathbf{p}) = \langle a^{\dagger}(\mathbf{p})a(\mathbf{p})\rangle_0$ and

$$\int_{\mathbf{x}}\mathrm{d}\mathbf{x}|\langle\left(\Phi^+ + \Phi^- - \langle\Phi\rangle_t\right)a(\mathbf{p})\rangle_t|^2 \sim \int \mathrm{d}\mathbf{p}'\frac{N_0(\mathbf{p}')N_0(\mathbf{p})}{2\omega_{\mathbf{p}'}}, \tag{54}$$

from which we find

$$\tau_R \sim \frac{1}{\lambda^2\int \mathrm{d}\mathbf{p}' N_0(\mathbf{p}')/(2\omega_{\mathbf{p}'})}. \tag{55}$$

As in the harmonic oscillator case, it is the third term on the right side of equation (53) that leads to variance reduction for macroscopic energy scales. The reduction time is inversely proportional to the total number of excitations in all modes. This will lead to rapid reduction for large scale excitations. Each mode tends towards a coherent state. As this occurs, we expect that the field tends towards classical behavior.

D. Fermionic state reduction

Here we consider a fermionic field coupled to our proposed scalar field theory, the effect of which is to induce state reduction in the fermionic sector. To see how this works let us set λ to zero for now and consider an interaction Hamiltonian of the type

$$H_{\text{int}}(t) = \int_{\mathbf{x}} \mathrm{d}\mathbf{x} j(x)\varphi(x). \tag{56}$$

Here j is some Hermitian current operator associated with the fermionic matter field. From equation (35) we have

$$\begin{aligned} H_{\text{int}}(t) &= \int_{\mathbf{x}} \mathrm{d}\mathbf{x} j(x) \int \frac{\mathrm{d}\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} \left\{ \exp\left(\mathrm{i}\mathbf{p}\cdot\mathbf{x} - \mathrm{i}\omega_{\mathbf{p}}t\right) a(\mathbf{p}) + \exp\left(-\mathrm{i}\mathbf{p}\cdot\mathbf{x} + \mathrm{i}\omega_{\mathbf{p}}t\right) a^{\dagger}(\mathbf{p}) \right\} \\ &= \int \frac{\mathrm{d}\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} \left\{ j^{\dagger}(\mathbf{p},t) \exp\left(-\mathrm{i}\omega_{\mathbf{p}}t\right) a(\mathbf{p}) + j(\mathbf{p},t) \exp\left(\mathrm{i}\omega_{\mathbf{p}}t\right) a^{\dagger}(\mathbf{p}) \right\}. \end{aligned} \tag{57}$$

Furthermore, we can formally solve the Tomonaga equation to find

$$|\Psi(t)\rangle = \exp\left\{ -\mathrm{i} \int_0^t \mathrm{d}u H_{\text{int}}(u) \right\} |\Psi(0)\rangle. \tag{58}$$

Now suppose that the fermionic state undergoes some spatial transfer of charge such that a pulse of current occurs. If the fermionic state is a j-eigenstate, we have

$$|\Psi(t)\rangle = \exp\left\{ \int \mathrm{d}\mathbf{p} (\alpha(\mathbf{p},t) a^{\dagger}(\mathbf{p}) - \alpha^{*}(\mathbf{p},t) a(\mathbf{p})) \right\} |\Psi(0)\rangle, \tag{59}$$

where the complex number α is given by

$$\alpha(\mathbf{p},t) = -\mathrm{i} \int_0^t \mathrm{d}u \frac{j(\mathbf{p},u)}{\sqrt{2\omega_{\mathbf{p}}}} \exp(\mathrm{i}\omega_{\mathbf{p}}u), \tag{60}$$

and $j(\mathbf{p},t)$ is the current eigenvalue at time t. Using the commutation relations for the creation and annihilation operators, and assuming that the initial φ state is unexcited, we find

$$\begin{aligned} a(\mathbf{p}')|\Psi(t)\rangle &= a(\mathbf{p}') \exp\left\{ \int \mathrm{d}\mathbf{p} (\alpha(\mathbf{p},t) a^{\dagger}(\mathbf{p}) - \alpha^{*}(\mathbf{p},t) a(\mathbf{p})) \right\} |\Psi(0)\rangle \\ &= \exp\left\{ \int \mathrm{d}\mathbf{p} (\alpha(\mathbf{p},t) a^{\dagger}(\mathbf{p}) - \alpha^{*}(\mathbf{p},t) a(\mathbf{p})) \right\} (a(\mathbf{p}') + \alpha(\mathbf{p}',t)) |\Psi(0)\rangle \\ &= \alpha(\mathbf{p}',t)|\Psi(t)\rangle. \end{aligned} \tag{61}$$

The final state is a φ-coherent state with eigenvalue α (cf. section 3.4 in [20]). This demonstrates that coherent states in φ are associated with j-eigenstates in the matter field. Reduction to a φ-coherent state should therefore induce reduction to a j-eigenstate in the fermionic sector.

It is tempting to associate φ with a gauge field such as the photon field or some proposed graviton field. The current j would then relate to a conserved charge, e.g. electric charge or energy-momentum. Such charge densities are a natural description of macroscopic observables.

IV. CONCLUSIONS

The key advance of this paper has been to develop a relativistic model of quantum state reduction which does not suffer from the infinite rates of energy density increase seen in previous proposals. We have outlined a model requiring just one extra parameter in addition to those of standard quantum theories in order to simultaneously describe the quantum behavior of individual excitations and the definite behavior of macroscopic objects.

In our approach, by having no coupling between the classical stochastic field and the particle creation operator, we ensure that the evolution equation cannot randomly create particles from the vacuum. Our model features only a coupling between the stochastic field and the particle annihilation operator. This is appealing for two further reasons. First, it leads to a reduction to coherent states. As coherent states saturate the bound of the Heisenberg uncertainty relation they make an obvious choice as a quantum counterpart to an idealized classical state. Second, by applying this mechanism to a bosonic field coupled to a fermionic field, we can induce state reduction to some charge density basis in the fermionic sector.

The ideas presented in this paper could be applied to the photon field or to a proposed graviton field in order to see state reduction to a conserved electric charge or energy-momentum basis in the associated matter fields. Since the model predicts an energy loss which could be significant in high-density highly accelerating matter environments, there may be the possibility of experimental investigation, e.g. by looking at the decay of high intensity electromagnetic waves or through the detection of gravitational waves.

Acknowledgements

The author wishes to thank D. Brody for valuable discussions.

---

[1] P. Pearle, Intl. J. Theo. Phys. **18**, (1979) 489.
[2] N. Gisin, Phys. Rev. Lett. **52**, (1984) 1657.
[3] G.C Ghirardi, A. Rimini, & T. Weber, Phys. Rev. **D34**, (1986) 470.
[4] L. Diósi, J. Phys. **A21**, (1988) 2885.
[5] G.C. Ghirardi, P. Pearle, & A. Rimini. Phys. Rev. **A 42**, (1990) 78.
[6] A. Bassi & G.C. Ghirardi, Phys. Rept. **379** (2003) 257.
[7] P. Pearle, in: *Open Systems and Measurement in Relativistic Quantum Field Theory*, H. P. Breuer and F. Petruccionne eds., Springer Verlag (1999).

[8] L. P. Hughston, Proc. Roy. Soc. **A452**, (1996) 953.
[9] D. C. Brody *et al.*, J. Phys. **A39**, (2006) 11029.
[10] S. L. Adler *et al.*, J. Phys. **A34**, (2001) 8795.
[11] D. C. Brody & L. P. Hughston, J. Phys. **A39**, (2006) 833-876.
[12] S. L. Adler, `quant-ph/0605072`.
[13] P. Pearle, in: *Sixty-Two Years of Uncertainty: Historical, Philosophical, and Physics Inqiries into the Foundations of Quantum Physics*, A. I. Miller ed., Plenum Press, New York (1990).
[14] G.C. Ghirardi, R. Grassi, & P. Pearle, Found. Phys. **20** (1990) 1271.
[15] S. L. Adler & T.A. Brun, J. Phys. **A34**, (2001) 4797-4809.
[16] P. Pearle, Phys. Rev. **A59**, (1999) 80-101.
[17] O. Nicrosini and A. Rimini, "Relativistic Spontaneous Localisations: a Proposal", preprint `quant-ph/0207145`.
[18] R. Tumulka, J. Statist. Phys. **125** (2006) 821.
[19] P. Pearle, Phys. Rev. **A71** (2005) 032101.
[20] C. C. Gerry & P. L . Knight, *Introductory Quantum Optics*, Cambridge Univ. Press (2005).
[21] S. Tomonaga, Prog. Theo. Phys. **1**, (1946) 27.
[22] J. Schwinger, Phys. Rev. **74**, (1948) 1439.
[23] G.C. Ghirardi, Found. Phys. **30**, (2000) 1337.
[24] F. Dowker & I. Herbauts, Class. Quant. Grav. **21**, (2004) 2963.
[25] F. Dowker & I. Herbauts, Found. Phys. Lett. **18**, (2005) 499.
[26] J. Conway & S. Kochen, Found. Phys. **36**, (2006) 1441.
[27] R. Tumulka, *"Comment on the 'Free Will Theorem'"*, J. Found. Phys., to be publ.
[28] A. Bassi & G.C. Ghirardi, *"The Conway-Kochen argument and relativistic GRW models"*, J. Found. Phys. to be publ.
[29] J. Conway & S. Kochen, `quant-ph/0701016`.
[30] G. 't Hooft, `quant-ph/0701097`.